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ON THE DYNAMIC PROPAGATION OF CRACKS†

V. V. Bolotin

Moscow

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The universal approach [1–3] to problems of the stability and quasistatic growth of cracks in solids during cyclic loading and loading which slowly varies with time is generalized to dynamic crack propagation processes. The general equation of d'Alembert–Lagrange dynamics for systems with two groups of generalized coordinates, Griffiths coordinates and Lagrange coordinates, serves as the starting point. The parameters characterizing the shape and dimensions of the cracks from the first group, while parameters, with an accuracy to which the displacement fields in a body with cracks are given for specified values of the specified coordinates of the first group, form the second group. A system of equations is obtained which enables one to describe the dynamical behaviour of bodies with cracks for fixed crack parameters, the dynamic propagation of cracks and also the transition of a body with a cracks–load system from one state to another, that is, the start-up and arrest of dynamic cracks. The application of the method is illustrated using model examples.

Fracture, which is accompanied by the dynamic propagation of cracks can arise both during the dynamic (in particular, impulsive loading) of bodies containing cracks or crack-like defects as well as when crack instabilities occur under conditions of quasistatic loading. A typical example is the final fracture of a structural element or component when a fatigue crack of a critical size has been formed. From a practical point of view, the greatest interest lies in establishing the conditions for the initiation of the dynamic growth of cracks and the conditions for the arrest of this growth. This is necessary for the well founded specification of standards regarding the permissible degree of defectiveness of structures which are subjected to the action of dynamic loads and for the choice of methods of preventing or retarding dynamic growth which has already begun.

A review of the results on dynamic problems in fracture dynamics can be found in [4–6]. The three types of theoretical investigations which are the most developed are: the determination of the fields of the dynamic stresses in bodies containing fixed cracks, the investigation of the processes accompanying the dynamic propagation of cracks with constant (specified) velocities and self-similar problems of the dynamic growth of cracks. The final result of the majority of investigations consists of the determination of the coefficients of the singular terms in the expressions for the stresses, that is, the determination of the dynamic stress intensity factors. By comparing these factors with certain crack stability characteristics, conclusions can be drawn concerning the conditions for the initiation and arrest of the dynamic growth of cracks.

The conditions for the start-up and arrest of cracks are among the least-studied problems of fracture dynamics. There are very few experimental data regarding these conditions and they are often contradictory [4–8] while theoretical investigations are only at an early stage. Boundary-value problems with fixed cracks and with dynamically propagating cracks are quite different. Here, the transition from the first type of problem to the second is non-trivial: the initial conditions for the start of the crack growth and the very instant at which start-up occurs must be determined from considerations which do not enter into the conditions of the problem during the first stage. The

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problem becomes more complex if account is taken of the dependence of the crack stability of the material on the rate of the crack growth.

The propagation of dynamic cracks in large-scale structures is accompanied by more or less pronounced oscillatory processes. As a rule, these processes, as well as the effect of secondary waves, are neglected. Meanwhile, experiments show that the rates of the growth of cracks in structures are relatively small compared with the rates of propagation of wave perturbations at, let us say, the Rayleigh velocity. The formation of longitudinal fractures in pipelines [9, 10] serves as an example. The velocity of sound in a gas is an order of magnitude lower than the rate of propagation of elastic waves in the walls of a pipeline. Furthermore, the progress of fracture fronts is regarded as a consequence of decompression, that is, the fall-off in the pipeline pressure. The rate of propagation of longitudinal fractures therefore turns out to be one or two orders of magnitude less than the velocity of elastic waves. Oscillatory effects give rise to the growth of nominal stresses on the cracks front. Moreover, the development of cracks can appreciably modify the rigidity of the system as a whole and, consequently, the reaction of a structure to the action of dynamic loads. On account of this, there is a need for a complex dynamical analysis of a structure under crack propagation conditions.

1. Let us now consider a "solid with cracks–load or a loading device" system with wide assumptions regarding the mechanical properties of the material of the body, the shape and dimensions of the cracks and also the nature of the loading. The shape and dimensions of the cracks are specified using m generalized Griffiths coordinates $l_1, \ldots, l_m$, the set of which is denoted by $\mathbf{l} = \{l_1, \ldots, l_m\}$. The number of generalized Griffiths coordinates is determined by the number of parameters, with an accuracy up to which the dynamically propagating cracks are specified. In the case of an open planar crack, its length serves as this parameter, that is, it is a single-parameter problem with respect to the generalized Griffiths coordinates. A planar crack of elliptical shape is specified using two generalized coordinates, the lengths of the semi-axes under the assumption that it retains its elliptical shape during the growth process. In the case of a planar crack of arbitrary shape in the plane, larger number of generalized coordinates have to be specified such as, for example, the set of lengths of the radius vectors drawn at different angles to points lying on the contour of the crack.

Confining ourselves to irreversible, "unhealing" cracks which are typical in structural materials, we choose the generalized Griffiths coordinates such that they are non-decreasing functions of time. The constraint conditions then take the form

$$\delta l_j \geqslant 0 \quad (j = 1, \ldots, m) \tag{1.1}$$

We shall assume that these constraints are ideal. The displacement vector field in the body for specified values of $l_1, \ldots, l_m$ is denoted by $\mathbf{u}(\mathbf{x}, t \,|\, \mathbf{l})$, where $\mathbf{x}$ is a coordinate vector and t is the time. The variation of the displacement field can be represented in the form of a sum

$$\delta \mathbf{u} = \delta \mathbf{u}_L + \delta \mathbf{u}_G \equiv \delta \mathbf{u}_L + \sum_{j=1}^{m} \frac{\partial \mathbf{u}}{\partial l_j} \delta l_j \tag{1.2}$$

where $\delta \mathbf{u}_L$ are the variations in the case of fixed cracks (Lagrangian variations) and $\delta \mathbf{u}_G$ are the variations which are generated by the change in the crack parameters (Griffiths variations). The variation of the Lagrangian displacements is illustrated in Fig. 1(a) while the combined Lagrangian and Griffiths variation is shown in Fig. 1(b).

Let us now apply the general equation of d'Alembert–Lagrange dynamics to a "body with cracks–load" system. Dynamical processes occur in the system such that the relationship

$$\delta A \equiv \delta A_e + \delta A_i + \delta A_f + \delta A_I \leqslant 0 \tag{1.3}$$

is satisfied at each instant of time.

Here δA_e and δA_i is the virtual work of the internal and external forces, respectively, δA_f is the virtual work of the forces resisting the propagation of cracks and δA_I is the virtual work of the inertial forces. As applied to the quasistatic growth of cracks $\delta A_I \equiv 0$ and we arrive at the

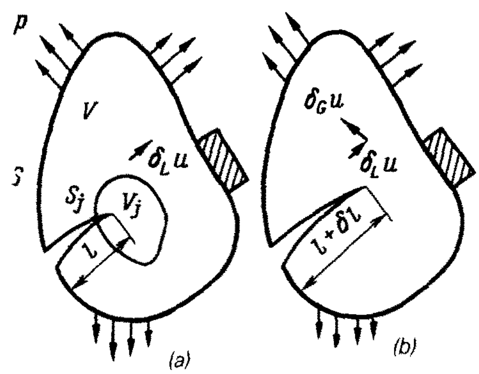

FIG. 1.

formulation of the problem from [1] where those states of the body are accepted for comparison which solely differ in the values of the generalized coordinates $l_1, \ldots, l_m$ when the equilibrium equations, the compatibility equation and the equation of the mechanical state are satisfied, as well as the boundary conditions and the Lyapunov stability conditions both in the investigated (unperturbed) state and in neighbouring ones. The "body with cracks–load" system is in equilibrium, if the sum of the components of the virtual work for any variations of the generalized Griffiths coordinates which satisfy conditions (1.1) is non-positive:

$$\delta A \equiv \delta A_e + \delta A_i + \delta A_f \leqslant 0 \tag{1.4}$$

Here, if $\delta A<0$ for any $\delta l_j>0$, the state of the "body with cracks–load" system is subequilibrium and stable. If, when $\delta l_j>0$, where $j=1, \ldots, m_1$, relationship (1.4) is satisfied with the equality sign and, in the case of the remaining $\delta l_j>0$, we have $\delta A<0$, then we say [1–3, 11] that the system is in an equilibrium state with respect to the generalized coordinates $l_1, \ldots, l_m$. This equilibrium state is stable when $\delta(\delta A)<0$ and unstable when $\delta(\delta A)>0$, where the second variation is also taken with respect to Griffiths. When $\delta(\delta A)=0$, the equilibrium state is neutral. This means that a given state is a boundary state between stable and unstable states or that, in order to reach a conclusion regarding the stability, it is necessary to investigate the following variations of δA.

In order to solve dynamic problems we return to the more-general relationship (1.3) containing the virtual work of the inertial forces. Here, we distinguish at least five cases: the motion of a body when there are fixed cracks, the beginning of the propagation (the start) of a crack, the motion of a body, which is accompanied by the dynamic propagation of a crack and the subsequent motion of a body when the crack front has arrested. If the number of crack parameters is greater than one, then more-complex situations are possible when cracks grow simultaneously or in turn, a crack starts to grow with respect to one of its parameters and subsequently starts to grow with respect to another parameter, etc. All of these situations are described by relationship (1.3). Taking account of the fact that the variations of the displacement field (1.2) are equal to the sum of two independent terms, we represent the virtual work in the form $\delta A=\delta_L A+\delta_G A$, where $\delta_L A$ is the work performed in Lagrangian variations and $\delta_G A$ is the work performed in Griffiths variations.

Let the motion of a body commence from a state in which all of the cracks are subequilibrium. Condition (1.4) then takes the form

$$\delta_L A = 0, \ \delta_G A < 0 \ (\delta l_k > 0, \ k = 1, \ldots, m) \tag{1.5}$$

The first condition is equivalent to the equations of motion and the natural boundary conditions while the second (the condition of subequilibrium) ensures the constancy of all of the generalized coordinates $l_1, \ldots, l_m$. If, at a certain instant of time t_*, the inequality $\delta_G A<0$ is replaced by the equality $\delta_G A=0$ for just one of the variations $\delta l_k>0$, this means that, when $t=t_*$, the necessary condition for the start of the crack growth along a generalized coordinate l_k is satisfied. The sufficient start-up condition has the form $\delta_G(\delta_G A)\equiv\delta_G^{\,2}A>0$ when $\delta l_k>0$ (the second variation is also carried out according to Griffiths). Trial calculations show that, in dynamic problems, the weaker conditions $\delta_G A = \delta_G{}^2 A = 0$ ensure the start of a crack.

Thus, the conditions for the start of cracks growth at a certain $t = t_*$, and simultaneously along m_1 generalized coordinates $l_1, \ldots, l_m$ can be represented in the form

$$\begin{gathered} \delta_L A = 0 \\ \delta_G A = 0, \ \delta_G{}^2 A \geqslant 0 \ (\delta l_k > 0, \ k = 1, \ldots, m_i) \\ \delta_G A < 0 \ (\delta l_k > 0, \ k = m_i + 1, \ldots, m) \end{gathered} \tag{1.6}$$

The equations which describe the dynamic propagation of cracks when $t > t_*$ along m_i generalized coordinates $l_1, \ldots, l_m$ with fixed values of $l_{m_i+1}, \ldots, l_m$ have the same form as (1.6). The growth of a crack along one of the coordinates (l_{m_i}, for example) ceases at the instant $t = t_{**}$ if the condition $\delta_G A = 0$ passes into the condition of subequilibrium: $\delta_G A < 0$ when $\delta l_m > 0$. Further motion occurs when conditions (1.6) are satisfied, where m_i is replaced by $m_i - 1$. After all the cracks have stopped growing, the motion of the body is described by relationships (1.5).

2. The conditions which establish the instant at which a crack (or cracks) starts leave open the question as to the velocity distribution immediately after the start of a crack. Intuition and tests suggest that the dynamic growth of a crack starts with a certain finite rate which differs from zero (in the numerical examples referring to experiment [6, 7], it is usual to put the initial velocity equal to $0.1\ c_s$, $0.2\ c_s$ and so on, where c_s is the rate of propagation of surface waves in the material). Similarly, if a crack appears as the result of the insertion of some object into the body, the rate of the crack growth is not the same in general as the corresponding velocity component on the leading edge of the object.

In order to find the velocity distribution at the instant when a crack starts $t = t_*$, let us consider the properties of the virtual work $\delta_G A$ in the neighbourhood of this instant. When $t = t_*$, let the motion begin along a generalized coordinate l_i. After the beginning of the motion, the condition $\delta_G A = 0$ is maintained, that is $\delta_G A(t_* + \Delta t) = 0$ when $\Delta t > 0$. Let $\delta_G A$ be a differentiable function of t and l_i in this neighbourhood. It follows from the condition $\delta_G A(t_*) = \delta_G A(t_* + \Delta t) = 0$ that, when $t = t_*$, we must have

$$d(\delta_G A)/dt = 0 \tag{2.1}$$

If, when $t = t_* + 0$, the virtual work $\delta_G A$ experiences a discontinuity which depends on the initial rate of the crack growth displacement then, instead of (2.1), we take the condition

$$\delta_G A\ (t_* + 0) = 0 \tag{2.2}$$

Since the rate of propagation of perturbations from the crack front is finite, the virtual work in (2.1) can be calculated when $t < t_*$, that is, using the displacement field found up to the start of the crack growth. Equations (2.1) and (2.2) are therefore the equations for determining the initial rate of the crack growth. In the case of simultaneous start-up along two or more coordinates, it is sufficient to make small changes to the initial data or other parameters of the problem in order to separate the instants of the starts of the crack growth. These relationships may also interpret both the conditions $\delta(\delta_G A) = 0$ at which the real displacements, associated with a small increment in t in the neighbourhood of $t = t_*$, are taken instead of the virtual displacements.

At the instant of the start-up of a crack, an easing of the corresponding unidirectional constraints occurs. This release of the constraints is, in general, accompanied by the appearance of shock forces. However, such forces do not arise in a continuous medium (at finite rates of propagation of perturbations). In the neighbourhood $t > t_*$ of the instant t_*, the virtual work from relationship (1.4) is equal to zero whence

$$\lim_{\Delta t \to 0} \int_{t_*}^{t_* + \Delta t} \delta A\, dt = 0$$

After integration, we arrive at a well-known equation [12] which relates the velocity discontinuities (both of points of the body as well as of the crack front) to the new parameters N_i, the generalized momenta of the shock forces:

$$\left[\frac{\partial T}{\partial(\partial l_i/\partial t)}\right]_{t_*}^{t_*+0} = N_i \tag{2.3}$$

Here T is the kinetic energy of the body with cracks.

Since the rate of propagation of perturbations is finite, the kinetic energy T is a continuous function of t in the neighbourhood of $t > t_*$. It follows from (2.3) that $T(t_* + 0) = T(t_*)$ so that $N_i = 0$ for all $i = 1, \ldots, m$. This conclusion does not preclude the possibility that the initial period of dynamic growth of cracks is accompanied by an increased resistance of the material. In order to describe the above-mentioned phenomenon, it is sufficient to introduce increased values of the cohesion forces (or of the specific breakdown work) on the initial segments. Here, additional characteristics of the material which, in the general case, depend on the rates of growth of cracks, enter into the treatment.

3. Difference and variational-difference approximations are used in computational fracture mechanics [4, 6]. Here, the number of generalized Lagrangian coordinates is chosen depending on the required accuracy of the calculations. In principle, such an approach enables one to construct accurate solutions of problems concerning the dynamics of bodies with cracks. In the case of bodies with finite dimensions, this is achieved by using the complete system of coordinate functions and procedures which ensure the convergence of the expansions in these functions. In the case of unbounded bodies, suitable integral transformations are taken instead of series. For simplicity, we shall henceforth assume that the number of generalized Lagrangian coordinates is finite, while allowing for the fact that this number may be taken to be very large in the numerical implementation. For instance, in calculations using the finite element method, the number of generalized coordinates (of modal point displacements) is of the order of 10^3, 10^4 or even greater.

Let us denote the generalized Lagrangian coordinates by $q_1, \ldots, q_n$ and the set of these coordinates by $\mathbf{q} = \{q_1, \ldots, q_n\}$ and represent the displacement field in the form

$$\mathbf{u}(\mathbf{x}, t \mid \mathbf{l}) = \sum_{k=1}^{n} q_k(t)\, \varphi_k(\mathbf{x} \mid \mathbf{l}) \tag{3.1}$$

where the coordinate vector functions $\varphi_k(x \mid \mathbf{l})$ satisfy the kinematic boundary conditions for a body with fixed cracks. To be specific, we shall assume that, unlike in the case of the generalized Griffiths coordinates l_j, no constraints are imposed on the signs of the variations δq_k. When account is taken of (1.2) and (3.1), we obtain the variations of the displacement field

$$\delta\mathbf{u}(\mathbf{x}, t \mid \mathbf{l}) = \sum_{j=1}^{m}\sum_{k=1}^{n} q_k \frac{\partial \varphi_k(\mathbf{x} \mid \mathbf{l})}{\partial l_j}\, \delta l_j + \sum_{k=1}^{n} \varphi_k(\mathbf{x} \mid \mathbf{l})\, \delta q_k \tag{3.2}$$

We will now formulate the equations of motion of a "body with cracks–load" system in terms of generalized forces. The components of the virtual work in relation (1.3) are linear forms of the variations δl_j and δq_k. In particular,

$$\delta A_e + \delta A_i = \sum_{j=1}^{m} G_j \delta l_j + \sum_{k=1}^{n} Q_k \delta q_k \tag{3.3}$$

where the factors G_j are analogous to the active generalized forces from [1–3, 11] (force promoting cracks) while the factors Q_k have the meaning of conventional generalized forces. If the material of the body is elastic, the external forces are potential ones and all of the constraints are ideal, then there exists a potential energy of the "body with cracks–load" system such that

$$G_j = -\partial \Pi / \partial l_j, \quad Q_k = -\partial \Pi / \partial q_k \tag{3.4}$$

We will represent the virtual work done against the forces resisting the growth of the crack in the form

$$\delta A_f = -\sum_i \int_{L_i} \gamma_i \,|\, d\mathbf{s}_i \times \delta\boldsymbol{\lambda}_i \,| \tag{3.5}$$

where γ_i is the specific fracture work per unit area of the crack (the new area of the surface is not doubled), ds_i is an element of length of the contour of the ith crack and $\delta\lambda_i$ is the vector of virtual displacement of the crack contour. Generally speaking, the values of γ depend on the positions and velocities of the crack fronts, that is, on l_j and dl_j/dt when $j = 1, \ldots, m$. The integration in (3.5) is carried out along the length L_i of the contour while the summation is carried out over all of the cracks in the body. When account is taken of (3.2), formula (3.5) reduces to the form

$$\delta A_f = -\sum_{j=1}^{m} \Gamma_j \delta l_j \tag{3.6}$$

where Γ_j are the generalized forces resisting the propagation of the crack front. We note that, generally speaking, the generalized forces G_j and Γ_j depend on the generalized Lagrangian coordinates, which are determined during the course of the solution.

Finally, let us consider the expression for the virtual work of inertial forces

$$\delta A_I = -\int_V \rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \delta\mathbf{u} \right) dV \tag{3.7}$$

where ρ is the density of the material and the integration is carried out over the entire volume of the body V. On substituting the expansion of the displacement field $\mathbf{u}(\mathbf{x}, t|\mathbf{l})$ from (3.1) and the variation $\delta\mathbf{u}(\mathbf{x}, t|\mathbf{l})$ from (3.2), we obtain

$$\delta A_I = \sum_{j=1}^{m} Y_j \delta l_j + \sum_{k=1}^{n} I_k \delta q_k \tag{3.8}$$

where Y_j and I_j are generalized forces of inertia.

If the dynamic processes are accompanied by the growth of cracks with respect to all of the generalized coordinates, then the system of $m+n$ second-order differential equations

$$G_j + Y_j = \Gamma_j, \; Q_k + I_k = 0 \; (j = 1, \ldots, m; \; k = 1, \ldots, n) \tag{3.9}$$

in the unknowns $l_1, \ldots, l_m, q_1, \ldots, q_n$ follows from (1.3), (3.3), (3.6) and (3.8). This system is solved with specified values of all the generalized Griffiths and Lagrangian coordinates as well as with specified values of their time derivatives at a certain initial instant of time. In a typical situation, the motion of the system begins from a state of rest with fixed crack fronts. The problem consists of finding the conditions for the start of crack growth, the paths and their rate of propagation, as well as the conditions for the arrest of the growth of cracks or the final fracture of the body.

4. We will now show how to overcome the difficulties associated with the transition from one system of characteristic equations to another. Let the initial state of the "body with cracks–load" system be subequilibrium with respect to all of the generalized Griffiths coordinates. Then, instead of (2.9), we have

$$G_j < \Gamma_j, \; Q_k + I_k = 0 \; (j = 1, \ldots, m; \; k = 1, \ldots, n) \tag{4.1}$$

The first group of relationships (4.1) ensures that all of the cracks are fixed, while the second group describes the dynamical processes in a body with fixed cracks. Relationships (4.1) are equivalent to conditions (1.5). During the first stage, we solve the second group of equations under the specified initial conditions which are imposed on the generalized Lagrangian coordinates, while checking that the inequalities from (4.1) are satisfied at each step for all $j = 1, \ldots, m$.

Let the equality $\delta_G A = 0$ when $\delta l_i > 0$ be attained for the first time at a certain instant t_*. If condition (2.1) or (2.2) turns out to be simultaneously satisfied then, at the instant of time t_*, the system ceases to be stable with respect to the generalized coordinate l_i. The subsequent evolution of the system is described using the relationships

$$G_j < \Gamma_j,\ G_i + Y_i = \Gamma_i,\ Q_k + I_k = 0 \qquad (4.2)$$
$$(j = 1, \ldots, m,\ j \neq i;\ k = 1, \ldots, n)$$

of which the first group ensures the immobility of the cracks with respect to the remaining generalized Griffiths coordinates. The other relationships form a system of $n+1$ second-order differential equations in l_i, $q_1, \ldots, q_n$. We find the initial conditions at the instant t_* for the generalized Lagrangian coordinates using the results of the integration in the preceding time interval. The initial values of l_i is specified by the conditions of the problem while we find the initial value of the derivative dl_i/dt by solving the system of equations

$$G_i = \Gamma_i,\ \left(\frac{\partial}{\partial t} + \frac{dl_i}{dt}\frac{\partial}{\partial l_i} + \sum_{k=1}^{n}\frac{dq_k}{dt}\frac{\partial}{\partial q_k}\right)(G_i - \Gamma_i) = 0 \qquad (4.3)$$

instead of using the starting instant t_*.

The second of the equations (4.3) signifies that condition (2.1) is satisfied, that is, the maintenance of a neutral state with respect to a generalized coordinate l_i in a small neighbourhood $t > t_*$. If the work $\delta_G A$ undergoes discontinuities when $t = t_* + 0$ then, instead of (4.3), we take the following:

$$G_i(t_*) = \Gamma_i(t_*),\ G_i(t_* + 0) = \Gamma_i(t_* + 0) \qquad (4.4)$$

Since the generalized forces (in particular, the generalized resistive forces) depend on the rate of growth of the corresponding cracks, Eqs (4.4) determine both the instant at which a fracture starts along a coordinate l_i as well as its initial velocity.

The conditions for the starting of cracks along the other generalized Griffiths coordinates are established in an analogous manner. Here, the inequalities from system (3.9) are replaced by the corresponding equalities which are added to the equations which have been integrated in the preceding step. For instance, if cracks grow along m_1 generalized coordinates $l_1, \ldots, l_m$ where $m_1 < m$, then, instead of (4.2), we solve the system $G_j + Y_j = \Gamma_j$, $Q_k + I_k = 0$ when $j = 1, \ldots, m$: $k = 1, \ldots, n$ and when the inequalities $G_j < \Gamma_j$ $(j = m_1 + 1, \ldots, m)$ are satisfied. The growth of a crack along any of the generalized coordinates ceases only when the corresponding condition $G_i + Y_i = \Gamma_i$ is violated. After growth has ceased along a generalized coordinate, the equation in this coordinate is excluded from the combined system of equations and replaced by the inequality $G_i < \Gamma_i$ which ensures the immobility of the crack. In general, the conditions $G_j + Y_j \leq \Gamma_j$ and $dl_j/dt \geq 0$ when $j = 1, \ldots, m$ must be checked throughout the whole of the process of solving the problem.

5. As model examples, we shall consider certain "beam" problems. Problems of this kind have been extensively discussed in fracture mechanics, starting from the work of I. V. Obreimov (1930). A review of some problems can be found in [13].

Let us assume that a thin elastic beam (or plate under conditions of cylindrical bending) is ripped off from an absolutely rigid mounting (Fig. 2a). We shall assume that the ratio h/l of the thickness of the beam h to the length l of the delaminating part and, also, the ratio f/l of the maximum deflection f to the length l are small compared with unity. We assume that the beam is clamped in the cross-section $x = l$ and loaded at the free end $x = 0$ with a force $P(t) = P_0F(t)$, where $P_0 = \text{const}$ and the function $F(t) \equiv 0$ when $t < 0$. The strength of the connecting layer is characterized by a specific fracture work γ, that is, the amount of work that has to be done in order to push forward the delamination of the beam by unit area. It is initially assumed that the specific work γ is independent of the rate of propagation of the crack dl/dt. We will assume that the connecting layer is exceedingly thin and we shall neglect the deformation of the part of the beam located at $x > l$.

Let the mass of the system M be concentrated on the end of the beam $x = 0$. As the generalized Lagrangian coordinate, we will take the deflection of the end, f. The length of the delamination l serves as the other generalized coordinate. For the deflection of the beam $w(x, t)$ when $0 \leq x \leq l$, we adopt an approximation of the type (3.1):

$$w(x,\ t \mid l) = f(t)\,[1 - (x/l)]^3 \qquad (5.1)$$

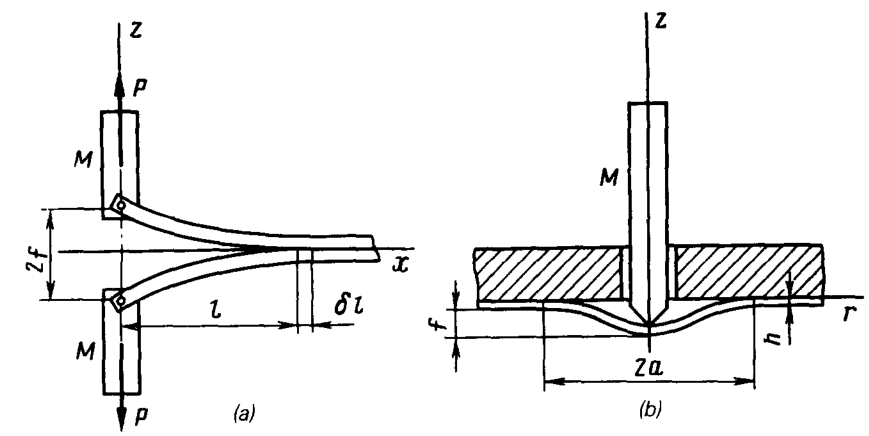

FIG. 2.

Formula (5.1) means that the dynamic deflection of the beam is taken to be similar to the static deflection with a time varying delamination length $l(t)$ and a displacement at the end $f(t)$. It is assumed that a crack is irreversible, that is, $dl/dt \geqslant 0$. We do not impose the constraint $f \geqslant 0$ since the analysis is terminated after the "collapse" of the delamination. Of course, a model with two degrees of freedom cannot include wave processes. However, it does enable one to illustrate the formulation and the procedure for solving typical problems concerning the initiation, dynamic growth and arrest of cracks.

The expression for the potential energy of deformation of the beam has the form $U = \frac{3}{2}Bf^2/l^3$, where B is the bending stiffness of the beam. For example, in the case of a beam of width b, we have $B = Ebh^3/12$, where E is Young's modulus.

We will now represent the sum of the virtual work done by external and internal forces in the form $\delta A_e + \delta A_i = -\delta U + P\delta f \equiv G\delta l + Q\delta f$. Then,

$$G = \tfrac{9}{2}Bf^2/l^4, \quad Q = P - 3Bf/l^3 \tag{5.2}$$

The virtual work done by the forces resisting the growth of delamination $\delta Af = -\gamma b\delta l$, whence $\Gamma = \gamma b$. In the case of a generalized inertial force, we have the expression $I = -M(d^2f/dt^2)$ while $Y \equiv 0$. Hence, instead of (4.1) and (4.2), we obtain

$$\tfrac{9}{2}Bf^2/l^4 \leqslant \gamma b, \quad M d^2f/dt^2 + 3Bf/l^3 = P_0F\,(t) \tag{5.3}$$

The inequality sign in the first relation corresponds to a fixed (subequilibrium) crack. If this inequality is satisfied, the second relationship is integrated with $l = \text{const}$. In particular, if the systems finds itself in a subequilibrium state when $t = 0$, then $l = l_0$, where l_0 is the initial size of the delamination. The point at which the equality sign is first achieved corresponds to the start-up of the crack. The instant t_* at which motion starts and the initial velocity can be found from Eqs (4.3). Taking account of formulas (5.2) these equations can be written in the form

$$\tfrac{9}{2}Bf^2/l^4 = \gamma b, \quad dl/dt = \tfrac{1}{2}\,(l/f)\, df/dt \tag{5.4}$$

Let the mass at the end of the delamination be $M \sim N\rho bhl$, where N is a numerical factor. Then from the second equation (5.4) we obtain the estimate $(dl/dt)_* \sim N^{-1/2}(h/l)c_0$, where $c_0 = (E/\rho)^{1/2}$ is the rate of propagation of longitudinal waves. For $h/l \ll 1$, $N \gg 1$ the rate of the crack growth is significantly less than the rate of wave propagation. Thus the model is internally non-contradictory.

For a more detailed discussion of the numerical results, we will change to dimensionless variables

$$\varphi = \frac{f}{f_0}, \quad \lambda = \frac{l}{l_0}, \quad \tau = \frac{\omega_0 t}{2\pi}, \quad \beta = \frac{f_0}{f_\gamma}$$

$$\left(f_0 = \frac{P_0 l_0^3}{3B}, \quad f_\gamma^2 = \frac{2\gamma b l_0^4}{9B}, \quad \omega_0^2 = \frac{3B}{Ml_0^3}\right)$$

Here f_0 is the static deflection of the end of the beam under the action of a force P_0, f_γ is the critical value of the static deflection and ω is the characteristic frequency of the beam (all the above-mentioned parameters are taken for the initial value $l = l_0$). The dimensionless parameter β characterizes the ratio of the force P_0 (or the corresponding quasistatic deflection) to the critical value with respect to the stability condition when there is a quasistatic load and when the length l_0 is fixed.

In dimensionless variables, relationships (5.3) take the form

$$\beta\varphi/\lambda^2 \leqslant 1, \ \varphi'' + 4\pi^2\varphi/\lambda^3 = 4\pi^2 F(\tau) \tag{5.5}$$

where differentiation with respect to the dimensionless time τ is denoted by primes. To find the instant of start-up τ_* and the initial rate $\lambda_*' \equiv \lambda'(\tau_*)$, instead of (5.4), we obtain the system of equations

$$\beta\varphi = 1, \ \varphi' - 2\varphi\lambda_*' = 0 \tag{5.6}$$

Let $F(\tau) = 1$ when $0 \leqslant \tau \leqslant \tau_1$ and $F(\tau) = 0$ when $\tau > \tau_1$. The motion starts from a state of rest $\varphi(0) = \varphi'(0) = 0$, $\lambda(0) = 1$, $\lambda'(0) = 0$. During the initial stage, when $\lambda = 1$, $\beta\varphi < 1$, the solution of the second equation of (5.5) has the form $\varphi = 1 - \cos 2\pi\tau$. When $\tau_* > \tau_1$ we obtain $\varphi = \cos 2\pi(\tau_1 - \tau) - \cos 2\pi\tau$. We find the instant of start-up τ_* and the initial rate λ_*' by solving the system of equation (5.6). Initially, it is more convenient to express β in terms of τ_* using the first equation. Then,

$$\beta = \begin{cases} (1 - \cos 2\pi\tau_*)^{-1}, & \tau_* < \tau_1 \\ [\cos 2\pi(\tau_1 - \tau_*) - \cos 2\pi\tau_*]^{-1}, & \tau_* > \tau_1 \end{cases} \tag{5.7}$$

after which the second of the equations (5.6) yields

$$\lambda_*' = \begin{cases} \pi\beta \sin 2\pi\tau_*, & \tau_* < \tau_1 \\ \pi\beta [\sin 2\pi(\tau_1 - \tau_*) + \sin 2\pi\tau_*]^{-1}, & \tau_* > \tau_1 \end{cases} \tag{5.8}$$

We do not write out the analytical formulas for the instant of arrest of a crack τ_{**} and its final size $\lambda_{**} \equiv \lambda(\tau_{**})$.

Certain numerical results are shown in Fig. 3, where the dimensionless time τ_0 prior to the start of growth of a crack and the dimensionless initial rate λ_*' are plotted as a function of the loading parameter β and the dimensionless duration of the impulse τ_1. Curves 1–4 correspond to the values $\tau_1 = 0.1, 0.2, 0.3, 0.5$. The value $\beta = ½$ corresponds to the load at which the dynamic deflection attains a critical value solely at the instant of maximum deflection of the beam. In the case of shorter impulses, a higher level of loading is required for the start-up. The curves $\lambda_*' = \lambda_*'(\beta)$ in Fig. 3 resemble the experimental dependences between the stress intensity factor (the parameter β may serve as its analogue in this case) and the rate of propagation of the crack [4, 5].

Figure 4 illustrates the change with time of the dimensionless deflection φ (the curves with a single maximum), the dimensionless length λ (the quasimonotonic curves) and the dimensionless rate of growth λ' of the delamination. Here, it is assumed that $\tau_* = ½$, that is the duration of the action of the force is half the period of the free vibrations of the beam of initial length. Curves 1, 2 and 3 are constructed for $\beta = 1, 1.25$ and

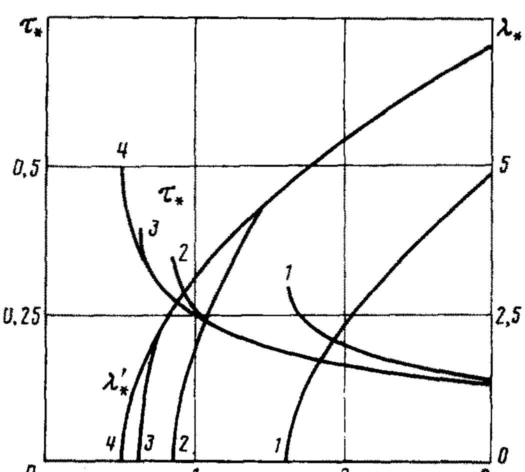

FIG. 3.

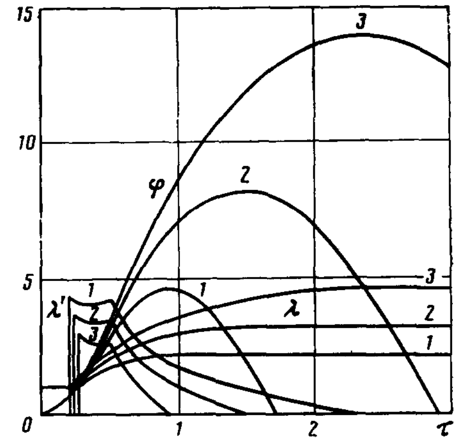

FIG. 4.

1.5, respectively. A crack is started when the dimensionless deflection φ attains the critical value $\varphi_* = \beta^{-1}$. The instant at which the crack is arrested τ_{**} and its final length λ_{**} also depend substantially on the level of the load, which is specified using the parameter β. The second maximum on the curves $\lambda' = \lambda'(\tau)$ corresponds to the instant $\tau = \tau_1$ when the action of the external force ceases.

6. As a second example let us take a thin elastic plate which is fastened to an absolutely rigid mounting and has circular layer separation from the top. An impact with a load M with an initial collision velocity V_0 (Fig. 2b) is performed at the centre of the layer separation at an instant $t = 0$. When $t<0$, the layer separation has a planar circular shape from above with a radius a_0 and is unstressed. When $t>0$, the centre of layer separation acquires a deflection $f>0$ and the radius of the layer separation a can start to grow on account of the fracture of the boundary. We assumed that the mass of the part of the plate where the layer has separated is small compared with the mass of the load M and that the deflection of the plate f is small compared with the radius of the layer separation a. We will approximate the deflection of the delaminated layer using the expression

$$w(r, t \mid a) = \frac{P}{16\pi D}\left(a^2 - r^2 + 2r^2 \ln \frac{r}{a}\right) \tag{6.1}$$

which satisfies the boundary conditions for a plate which is clamped along a circular contour $r = a$. The specific fracture work γ is taken to be constant.

In this case, the virtual work of the external forces is $\delta A_e = Mg\delta f$ where g is the acceleration due to gravity (the load falls vertically) and the virtual work of the internal forces $\delta A_i = -\delta U$, where U is the potential bending energy of the delaminated layer. When account is taken of (6.1), within the framework of the linear theory of the bending of a plate we have $U = 8\pi f^2 D/a^2$, where D is the cylindrical stiffness and $f = Pa^2/(16\pi D)$. The virtual work of the inertial forces $\delta A_I = -M(d^2f/dt^2)\delta f$, and the fracture work $\delta A_f = -2\pi a\gamma\delta a$, from which we find the generalized forces

$$G = 16\pi Df^2/a^3, \quad \Gamma = 2\pi a\gamma, \quad Q = Mg - 16\pi Df/a^2$$
$$I = -Md^2f/dt^2, \quad Y \equiv 0 \tag{6.2}$$

Relation (4.1) and (4.2), as in the preceding example, lead to a differential equation in the generalized Lagrangian coordinate f and an inequality in the generalized Griffiths coordinate l. When account is taken of formulas (6.2), we obtain

$$16\pi Df^2/a^3 \leqslant 2\pi a\gamma, \quad Md^2f/dt^2 + 16\pi Df/a^2 = Mg \tag{6.3}$$

The initial conditions have the form $a(0) = a_0(0) = 0$, $f(0) = 0$, $f'(0) = V_0$, where V_0 is the velocity of the load at the instant of impact.

Let us now introduce the dimensionless variables

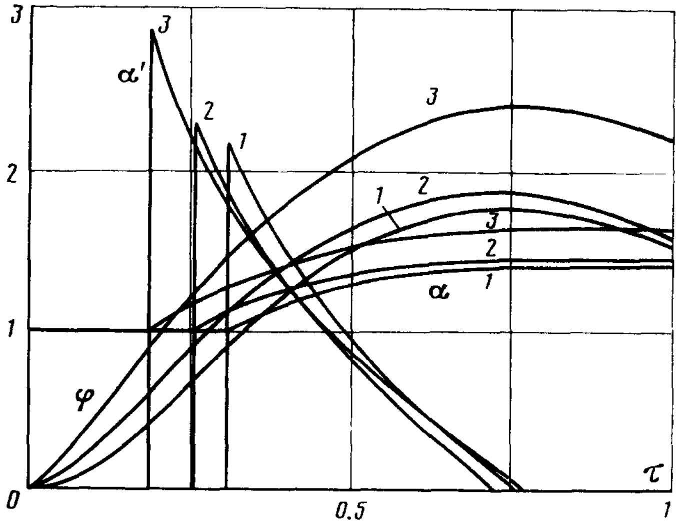

FIG. 5.

$$\varphi = \frac{f}{f_0}, \quad \alpha = \frac{a}{a_0}, \quad \tau = \frac{\omega_0 t}{2\pi}, \quad \beta = \frac{f_0}{f_\gamma}$$

$$\left(f_0 = \frac{Mga_0^2}{16\pi D}, \quad f_\gamma^2 = \frac{a_0^4\gamma}{16D}, \quad \omega_0^2 = \frac{16\pi D}{Ma^2} \right)$$

Here, f_0 has the meaning of the static deflection of the delaminated layer with an initial radius a_0 under a force Mg and f_γ is the equilibrium value of this deflection, that is, the value at which the detached layer of radius a_0 can start to grow even during quasistatic loading by a force Mg. Additionally, a characteristic vibrational frequency ω_0 is introduced for the layer delaminated with a radius a_0 and a mass M which is concentrated at the centre. As a result, instead of (6.3), we arrive at the relationships

$$\beta\varphi \leqslant \alpha^2, \; \varphi'' + 4\pi^2\varphi/\alpha^2 = 4\pi^2 \tag{6.4}$$

with the initial conditions $\alpha(0) = 1$, $\alpha'(0) = 0$, $\varphi(0) = 0$ and $\varphi'(0) = \varphi_0'$, where $\varphi_0' = 2\pi V_0/(\omega_0 f_0)$.

While $\beta\varphi(\tau) < 1$, the delaminated layer does not increase. Since, for any $V_0 \geqslant 0$ and $a = a_0$, the maximum deflection under the load is not less than $2f_0$, the growth of delaminated layers takes place when the loading parameter $\beta > ½$. This growth starts at $\tau = \tau_*$, where τ_* is a root of the equation $\beta\varphi(\tau) = 1$ and $\varphi(\tau)$ is the solution of the second equation of (6.4) when $\alpha = 1$. Calculations led to the equation $1 - \cos\tau_* + \varphi_0'\sin\tau_* = 1/\beta$. The initial rate of growth $\alpha_*' \equiv \alpha'(\tau_*)$ of the delaminated layers is found from the condition $\beta\varphi'(\tau_*) = 2\alpha'(\tau_*)$. Whence, $\alpha_*' = \beta(\sin\tau_* - \varphi_*'\cos\tau_*)$, etc.

The change in the dimensionless time τ, the dimensionless values of the radius of the delaminated layer $\alpha(\tau)$, the rates $\alpha'(\tau)$ as well as the deflection at the centre $\varphi(\tau)$ are shown in Fig. 5 for the following values of the initial rate of loading parameter: $\varphi_0' = 0, 2, 5$ (curves 1, 2 and 3, respectively). On conversion to dimensional variables with respect to the impact velocity V_0, these values are extremely reasonable so that Fig. 5 illustrates the behaviour of the delaminated layer under a low impact velocity. As the impact velocity is increased, the time τ_* up to the start of the growth of the crack increases and the duration of the growth stage $\tau_{**} - \tau_*$ also becomes longer. Calculations reveal only an insignificant breakdown of the monotonic dependence of the instant of arrest on the collision velocity (Fig. 5).

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THE INTRODUCTION OF "FORBIDDEN" AMPLITUDES WHEN CALCULATING THE WAVE RESISTANCE OF A SHIP†

E. L. AMROMIN, A. N. LORDKIPANIDZE and YU. S. TIMOSHIN

St Petersburg

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"Forbidden" values of the amplitudes are introduced into the Havelock formula, used in the linear theory of ship waves, which associates the wave resistance with their amplitudes. As a result, it is possible to achieve satisfactory agreement between calcaulation and experiment for various shapes of vessels.

THE WAVE resistance of a vessel depends on the amplitude of the ship waves caused by the vessel. Linear theory assumes that these amplitudes are directly proportional to the intensities of the wave-forming features by which bodies moving close to the free surface are replaced and this enables one to obtain relatively simple formulas for calculating the wave resistance of a ship R_w [1, 2]. However, as the above-mentioned intensities are increased, the experimental dependences for the amplitudes deviate so strongly from linear dependences that the divergence between theory [1, 3] and experiment for R_w turns out to be striking. Attempts to solve the non-linear spatial problem of ship waves both by expanding the flow characteristics in series in powers of a small parameter [4] as well as by using characteristics distributed over its boundaries do not yield satisfactory results in the calculation of R_w for ships of different shapes and different values of the Froude number, Fr, in spite of the considerable computer resources which are used.

In this situation, it is reasonable to appeal to what is probably the simplest method of partially taking non-linearity into account, that is, to the introduction of limiting or "forbidden" amplitudes. Limiting amplitudes, which cannot be exceeded for any intensity of the perturbation source, are encountered in various branches of mechanics and physics.

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